On-the-Job Search, Human Capital Accumulation and Endogenous Firm Productivity

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Abstract

This paper studies the implications of human capital accumulation for firms' decisions to invest in match-specific productivity. This brings a novel insight into the relationship between human capital accumulation and the equilibrium wage distribution. The paper extends a wage-posting model of on-the-job search with learning-by-doing (Burdett et al., 2011) by introducing an investment choice in the firms' problem. Since high-paying firms see their workers quit less often and attract more experienced workforce in equilibrium, they invest more in productive capacity; as they are more productive, they pay higher wages. This links the rate of human capital accumulation to the equilibrium distribution of firm productivities and wages. The model is solved numerically and it is shown that high rates of accumulation imply more disperse and positively skewed offer distributions, and have a quantitatively large effect.

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1 Introduction

When workers' productivity increases with experience, ex-ante identical firms in a frictional labour market characterised by churning have differential pay policies, not only because higher pay retains workers for longer but also because it implies that firms attract more experienced work-force in equilibrium (Burdett et al., 2011). It has been long recognized that the relationship between pay policies and turnover provides incentives for firms to also differentiate themselves in terms of productivity (Mortensen, 2000; Acemoglu and Shimer, 2000; Quercioli, 2005; Mortensen, 2005). Intuitively, a firm whose workers are less likely to quit gains by investing in the productivity of the match (or in the workers' human capital) implying positive relationship between firm productivity and wage offers. The possibility of endogenous determination of productive differentials across firms is interesting because empirically these differentials are large and persistent (Bartelsman and Doms, 2000; Lentz and Mortensen, 2008) and essential for improving the fit of standard models of wage dispersion (Postel-Vinay and Robin, 2002).

This paper studies firms' pay policies and decisions to invest in match-specific productivity in a labour market where workers accumulate human capital through experience and search randomly. Since high-paying firms see their workers quit less often and also attract more experienced workforce they have incentive to make larger productivity investments; given that their productivity is higher (both match productivity and average productivity of workers) they tend to pay higher wages. Such feedback links the rate of human capital accumulation and the properties of the technology transforming investment into productivity gains to the equilibrium distribution of pay policies, firm productivities, and wages.

To address the question I present a steady-state model of frictional labour market featuring on-the-job search, human capital accumulation, matching and job-creation, and a firm-level decision to invest in productive capacity. The environment is closely related to Burdett et al. (2011) and, in particular, the supply-side of the labour market is identical to the more recent generalization by Carrillo-Tudela (2012). Workers enter the market with different ex-ante abilities, accumulate human capital exponentially while employed, search randomly for employment when unemployed and for better-paying employment when employed. As in Carrillo-Tudela (2012) meeting rates of employed and unemployed workers differ by a constant fraction, but in addition depend on equilibrium market tightness. Firms create "job sites" which at any point in time are either vacant or filled[1]. Upon entry a vacancy commits to a time-invariant job-specific investment, which determines its productivity once matched and flow cost while vacant, and piece-rate[2] paid to any employee following a match. I characterize the steady-state equilibrium and, in particular, show that higher-paying firms invest more in match-specific productivity. I then parameterize the model consistently with Carrillo-Tudela (2012) and investigate numerically how the rate of human capital accumulation and the parameters related to investment affect equilibrium dispersion of productivity and piece-rate offers. The results imply that high rates of accumulation and high returns to investment imply more disperse and positively skewed offer distributions, and have quantitatively large effect.

The labour market environment in this paper is most closely related to Carrillo-Tudela (2012) and Burdett et al. (2011) but differs insofar as I focus on endogenous productivity dispersion and job-creation. Firms' decision to invest in productivity at the point of posting a vacancy closely resembles the setup in Mortensen (2000) but there workers do not accumulate human capital. By merging these two ideas, the analysis here contributes to the literature by emphasising the interaction between human-capital accumulation, pay policies, and productivity investment in the determination of equilibrium distribution of earnings and firm productivity. To the extent that growth rates of human capital (i.e. the steepness of the learning curve) differ across labour-market segments[3], wage and productivity dispersion

[1]As I abstract from issues of firm size, the term "firm" refers to one of these job sites.

[2]As human capital accumulation implies that the productivity of a match changes over time, modelling explicitly how the total product is divided between worker and firm is challenging. The assumption of piece-rate offers (also used in other studies such as Bagger et al. (2014) and Fu (2011)) circumvents the difficulty.

[3]For example, Bagger et al. (2014) show that it differs substantially by education, with most educated workers observing largest growth rates.

will also differ. Furthermore, if the rates of human capital accumulation or the cost/return to investment in productivity for firms can be affected by policy, the analysis here presents a framework for evaluating its distributional consequences.

On the other hand, my emphasis on the interaction between human capital accumulation, pay policies and investment in productivity, is similar to Fu (2011) but the analysis here differs in two important respects. First, I model human capital accumulation as universal (learning-by-doing) while Fu (2011) emphasises its emergence as a consequence of costly firm-specific training decision. Second, I model investment in productivity as match-specific, while in her model a firm investing in training increases its workers' general human capital. While the main question of interest is very similar, the underlying assumptions are, in a sense, diametrically opposed.

The paper proceeds as follows. Section 2 formulates the model. Sections 3 and 4 state the optimisation problem of workers and firms respectively. Section 5 defines steady-state equilibrium and the latter is characterised in Section 6. Section 7 presents the results from the numerical analysis. Section 8 concludes.

2 Environment

Time is continuous. A continuum of heterogeneous workers, normalized to unit mass, and a continuum of ex ante identical firms interact in a frictional labour market. All agents are risk-neutral and discount the future at rate, r. The system is in steady state.

Workers enter the labour market unemployed, with no experience, and with different initial productive abilities, summarised by the random variable, ϵ. At a constant Poisson rate, ϕ, a worker leaves the labour market for good. New workers enter the labour force at the same rate, implying that the distribution of ϵ among workers, $A(\epsilon)$, is time-invariant.

A worker's productivity is summarised by an individual-specific variable, y, determined by ability and experience. For a worker of ability ϵ who has been employed for x years

$$y(\epsilon, x) = \epsilon e^{\rho x}$$

Human capital is general, accumulates at a constant rate, ρ, during employment and does not depreciate. To prevent infinite accumulation, assume $\phi > \rho$.

Both employed and unemployed workers search. An unemployed worker meets a random vacancy at Poisson rate $\lambda_u(\theta)$, while an employed worker meets a random vacancy at Poisson rate $\lambda_e(\theta)$. The number of meetings in the economy is determined by a matching technology and market tightness, θ, is the ratio of vacancies to effectively searching workers[4]. I assume $\lambda_u > \lambda_e$ and $\lambda \equiv \lambda_e/\lambda_u$, the relative search intensity of employed as compared to unemployed workers, to be constant.

Firms create "job sites" (Mortensen, 2000) which at any point in time are either vacant or filled. As I abstract from issues of firm size, the term "firm" refers to one of these job sites. A firm enters the labour market by posting a vacancy, simultaneously chooses the productivity of its job opening and commits to paying a constant fraction $\tau \in (0, 1)$ of the future match product to any prospective employee. To set ideas, imagine that the firm opens a vacancy by investing in capital, k, and the investment maps into a unique time-invariant firm-specific productivity, $p(k)$. To capture the idea that investment is costly, assume that while still vacant, a more productive firm incurs a larger flow cost, $c(k)$, perhaps because of expenses necessary to preserve productive capacity when a job is still idle[5]. Assume $p'(.) > 0$, $p''(.) < 0$, $\lim_{k\to\infty} p'(k) = 0$, $c'(.) > 0$, and $c''(.) \geq 0$ (see below). A vacancy meets random workers at rate $\eta(\theta)$ and becomes a "job" upon meeting a worker who accepts the match. Jobs are destroyed exogenously at a Poisson rate, δ, or when the matched worker finds better-paid employment. Upon destruction the job becomes a vacancy (with the same amount of capital) while the worker becomes unemployed (upon exogenous destruction) or

[4]During most of Sections 3 and 4 explicit notational reference to market tightness is suppressed for brevity.
[5]Alternatively, one can think of $c(k)$ as the opportunity cost of not filling a vacancy.

transits to higher-paid employment. In equilibrium free entry drives the value of posting a vacancy to zero.

A firm of productivity $p(k)$ matched with a worker of productivity y produce flow of output $p(k)y$. The match product is shared according to the pre-specified piece rate, τ - at each point of time the worker receives flow wage $\tau p(k)y$ and the firm receives flow profit $(1-\tau)p(k)y$. For notational purposes, let $z \equiv \tau p(k)$. Assume that the flow of benefits to an unemployed worker is proportional to her productivity, $z_b = by$, where b is a policy-set parameter[6].

The flow of meetings between workers and vacancies is described by a matching technology, $m(u, 1-u, v)$, where u and v are the unemployment rate and the measure of vacancies. Employed and unemployed workers are assumed 1-to-λ perfect substitutes in the matching technology, that is $m(u, 1-u, v) = m(u+\lambda(1-u), v)$. Accordingly let $\theta = v/(u+\lambda(1-u))$ denote labour market tightness. In what follows, I assume directly a Cobb-Douglas matching function of the form

$$m(u+\lambda(1-u), v) = \beta(v)^{\alpha}[u+\lambda(1-u)]^{1-\alpha}$$

In order for the number of meetings prescribed by the matching function to equal the number of meetings accruing to workers, the meeting rates are related to market tightness and the matching parameters in the following way[7]

$$\begin{aligned}
\eta(\theta) &= m(1/\theta, 1) = \beta\theta^{\alpha-1} \\
\lambda_u(\theta) &= m(1, \theta) = \beta\theta^{\alpha} \\
\lambda_e(\theta) &= \lambda\lambda_u(\theta) = \lambda\beta\theta^{\alpha}
\end{aligned}$$

[6]This implies that unemployment benefits are proportional to a worker's productivity, rather than their most recent wage. The specification is preferred for analytical convenience - unemployment income proportional to past wages will imply heterogeneity of reservation pay rates among the pool of unemployed workers, an aspect from which the analysis here abstracts. For a discussion see Burdett et al. (2011)

[7]This is a standard specification of matching function in the context of on-the-job search; for example see Dolado et al. (2009)

3 Workers' Behaviour

This section formulates the dynamic problem faced by workers and characterises their optimal behaviour given any profile of firms' behaviours (summarised by the distribution of z across vacancies, $F(z)$) and labour market tightness. It is important to notice that given z workers have no preference over individual firms' combinations of payout policies (τ) and firm-specific productivities ($p(k)$). This property is convenient because in turn will imply that given z a firm's equilibrium turnover is independent of k which yields a tractable investment problem. For the same reason the workers' problem is identical to the one in Carrillo-Tudela (2012). For completeness, I state the workers' problem, its solution, and discuss the main intuition but do not state the proofs explicitly as they can be found in Carrillo-Tudela (2012).

Let $W^U(y|F(.),\theta)$ be the lifetime utility of an unemployed worker with productivity y given offer distribution $F(z)$ and tightness, θ; let $W^E(y,z|F(.),\theta)$ be the lifetime utility of a worker with productivity y employed at z and facing offer distribution $F(z)$. Recall that the offer distribution and tightness are determined in equilibrium.

An unemployed worker with productivity y faces random death risk, discounts the future, receives flow income by and meets firms at rate λ_u. Upon meeting a vacancy she decides whether to match based on a comparison of her expected lifetime utilities under unemployment and under employment at that firm. The Bellman equation for the value of unemployment is then

$$(r+\phi)W^U(y|.) = by + \lambda_u \int_{\underline{z}}^{\bar{z}} \max\left[W^E(y,z'|.) - W^U(y|.), 0\right] \partial F(z') \tag{1}$$

A worker with productivity y employed at a firm paying z faces random death and job destruction risks, discounts the future, receives flow income zy, accumulates human capital and meets new vacancies at rate λ_e. Upon meeting a new vacancy the worker compares her expected lifetime utilities under her current employer and under the one just met and chooses optimally (assuming that a worker never optimally quits into unemployment - which is true in equilibrium). Since working at higher piece rate is always better for the worker, the value of employment increases with z. Therefore, it is immediate that a worker quits a firm to match with another firm if and only if the new firm offers higher z. We also adopt the convention that if a worker is offered exactly her current z she remains with the incumbent firm. The Bellman equation for the value of employment is then

$$\begin{aligned}(r+\phi \quad + \quad \delta)W^E(y,z|.) = zy + \delta W^U(y|.) + \frac{\partial W^E(y,z|.)}{\partial y}\rho y + \\ + \quad \lambda_e \int_z^{\bar{z}} \max\left[W^E(y,z'|.) - W^E(y,z|.), 0\right] \partial F(z') \end{aligned} \tag{2}$$

Denote $q(z) \equiv \phi + \delta + \lambda_e(1 - F(z))$, the rate at which a worker employed at z leaves her employer. The optimal behaviour of a worker as implied by (1) and (2) is then completely characterised by the following

Claim 1 *Optimal workers' behaviour is fully described by a set of reservation values of z for employed and unemployed workers.*

1. *The reservation value of an employed worker is her current z.*

2. *The reservation value for an unemployed worker, z^R, satisfies*

$$\begin{aligned}(r+\phi)z^R(F(z),\theta) \quad &= \quad (r+\phi-\rho)b + \\ &+ \quad [\lambda_u(r+\phi-\rho) - \lambda_e(r+\phi)] \int_{z^R}^{\bar{z}} \frac{1-F(z')}{q(z')+r-\rho} \partial z'\end{aligned} \tag{3}$$

3. *Sufficient condition for existence and uniqueness of a solution is $\bar{z} > b(r+\phi-\rho)/(r+\phi)$*

Proof *See Proposition 1 in Carrillo-Tudela (2012).*

The main arguments behind the proof are the following. First, $W^U(y)$ is independent of z and $W^E(y,z)$ is increasing in z. Therefore, worker behaviour is indeed described by cutoff, or reservation values of z, at which she optimally changes states. Second, since all payoffs are proportional to y, then so are the value functions. That is, there exist real valued α^U and $\alpha^E(z)$ such that

$$W^U(y) = \alpha^U y, \text{ and } W^E(y,z) = \alpha^E(z)y$$

Proposition 1 can be then proved by substituting the latter back into the value functions, evaluating at z^R and solving the resulting system of equations. The proportionality of the value functions to y implies that the reservation value of an unemployed worker does not depend on her productivity. The latter is due to the fact that workers are paid in piece rates and that human capital accumulates at constant rate.

When there is no learning by doing, $\rho = 0$, and unemployed and employed workers search at the same intensity, $\lambda_e = \lambda_u$, (3) implies that $z^R = b$. When $\rho = 0$ and $\lambda_e < \lambda_u$, (3) implies that $z^R > b$, because by entering into employment a worker forgoes the opportunity to search at a higher intensity, hence making unemployment relatively more valuable. Positive rate of human capital accumulation, $\rho > 0$, makes employment relatively more valuable and, therefore, decreases the reservation value of z. At a sufficiently high ρ, the implied z^R may even become negative - workers might be willing to pay in order to be able to stay in employment and accumulate experience.

4 Firms' Behaviour

I now turn attention to the optimal behaviour of firms, taking the reservation value of workers, z^R, and market tightness, θ, as given. Let $J(z,k,\epsilon,x)$ be the value to a firm with capital, k, paying z from being matched with a worker with initial ability ϵ and experience x. Let $V(z,k)$ be the value to a firm from posting a vacancy with capital k committed to paying z.

A firm with capital k, paying according to z, matched with a worker of productivity y produces an instantaneous flow $p(k)y$ and receives flow profit $(p(k)-z)y = (p(k)-z)\epsilon e^{\rho x}$. It discounts the future, gets more productive over time due to human capital accumulation, and becomes vacant if exogenously destroyed or its worker quitted. Its value is therefore implicitly defined by the ordinary differential equation

$$(r+q(z))J(z,k,\epsilon,x) = (p(k)-z)\epsilon e^{\rho x} \quad + \quad \frac{\partial J(z,k,\epsilon,x)}{\partial x} + (\delta + \lambda_e(1-F(z)))V(z,k) \tag{4}$$

Solving (4) (integrating by parts with respect to x over the interval $[x,\infty)$, using an integration factor $e^{-(r+q)x}$ and assuming $\lim_{x'\to\infty} J(.,x')e^{-(r+q(.))x} = 0$) yields

$$J(z,k,\epsilon,x) = \frac{(p(k)-z)\epsilon e^{\rho x}}{q(z)+r-\rho} + \frac{\delta + \lambda_e(1-F(z))}{q(z)+r} V(z,k) \tag{5}$$

Direct examination then confirms that indeed $\lim_{x'\to\infty} J(.,x')e^{-(r+q(.))x} = 0$.

A vacancy that invested k and committed to pay according to z incurs a flow cost $c(k)$ and meets a random worker at Poisson rate $\eta(\theta)$. The value of a vacancy offering $\{z,k\}$ is then given by

$$(r+\eta(\theta))V(z,k) = -c(k) + \eta(\theta)M(z,k)$$

where $M(z,k)$, the expected value from a match, is an appropriately weighted average of the job value (see (12)). The firm chooses its $\{z,k\}$ upon entering the market so that its vacancy value is maximised. The Bellman equation for a vacancy is then

$$(r+\eta(\theta))V = \max_{z,k} \{-c(k) + \eta(\theta)M(z,k)\} \tag{6}$$

Given free entry, V is driven down to 0 in equilibrium.

5 Equilibrium

Let U^ϵ, N^ϵ(x), and $H^\epsilon(x, z)$ be the unemployment rate, the distribution of experience among unemployed workers, and the joint distribution of experience and z among employed workers of ability ϵ, respectively.

Definition A steady-state equilibrium is a set of pay and investment policy pairs, $Z \times K$, an offer distribution, $F(z)$, over Z, associated distribution of productivities, a reservation offer rate, z^R, unemployment rate, U^ϵ, steady state distributions, $N^\epsilon(x)$ and $H^\epsilon(x, z)$, and market tightness, θ, such that

- Given $F(z)$ and θ, the reservation offer rate, z^R, is given by (3); worker's behaviour is optimal.
- Given z^R, $F(z)$, and θ,

$$\{z, k\} = \arg\max_{z,k} V(z, k), \forall\{z, k\} \in Z \times K$$

  The behaviour of each firm is individually optimal.

- The distribution of offers, $F(z)$, is consistent with individual firms' optimal offer policies.
- U^ϵ, $N^\epsilon(x)$, and $H^\epsilon(x, z)$ are consistent with equilibrium turnover given optimal behaviour and market tightness.
- The market tightness, θ, is such that the free entry condition holds

$$0 = V \equiv V(z, k), \forall\{z, k\} \in Z \times K$$

6 Characterisation

6.1 Turnover and distributions

To characterise equilibrium, I first derive expressions for U^ϵ, $N^\epsilon(x)$, and $H^\epsilon(x, z)$. Notice that conditional on z, turnover is independent of firms' investment decisions.

Consider U^ϵ. The inflow of workers to this pool over a time interval of length dt is $(\phi + \delta(1 - U^\epsilon))dt$ (new labour-market entrants and previously employed workers whose jobs were destroyed) with associated outflow $(\lambda_u + \phi)dtU^\epsilon$ (previously unemployed workers who became employed or left the labour market). Equating the flows yields a steady-state unemployment rate

$$U \equiv U^\epsilon = \frac{\phi + \delta}{\phi + \delta + \lambda_u} \tag{7}$$

which is independent of ability, ϵ.

Consider $N^\epsilon(x)$, the pool of unemployed workers of type ϵ with experience below x. The inflow of workers to this pool is $(\phi + \delta(1 - U)H^\epsilon(x, \bar{z}))dt$ (new labour-market entrants or previously employed workers with experience below x whose jobs were destroyed) and the outflow is $(\phi + \lambda_u)dtUN_\epsilon(x)$ (this pool is only left through transitions to employment or non-participation, as experience is constant during an unemployment spell), yielding

$$N_\epsilon(x) = \frac{\phi(\phi + \delta + \lambda_u) + \delta\lambda_u H_\epsilon(x, \bar{z})}{(\phi + \lambda_u)(\phi + \delta)} \tag{8}$$

Consider $H_\epsilon(x, z)$, the pool of workers of type ϵ with experience no more than x earning at rate no more than z. The inflow to this pool is $UN_\epsilon(x)\lambda_u dt$ (previously unemployed workers of experience below x who found a job) and outflow

$$(1 - U)H_\epsilon(x, z)q(z)dt + (1 - U)(H_\epsilon(x, z) - H_\epsilon(x - dt, z)) + O(dt^2)$$

(the first term describes workers who transited to unemployment, non-participation or employment at rate above z; the second term describes those who remained employed at rate below z but accumulated experience above x; $O(dt^2)$ accounts for the fact that some workers both accumulated experience above x and left employment at pay rate below z, but this term is of order dt^2). Equating inflow to outflow and rearranging yields the first-order ordinary differential equation in $H_\epsilon(x,z)$

$$(\phi+\delta)N_\epsilon(x)F(z)=q(z)H_\epsilon(x,z)+\frac{\partial H_\epsilon(x,z)}{\partial x} \tag{9}$$

which can be solved by directly integration using factor $e^{q(z)x}$. By solving simultaneously (8) and (9) closed-form expressions for the steady-state distributions are obtained. Given θ and $F(.)$ all flows are determined by the worker's problem and therefore the steady-state distributions are identical to the ones in Carrillo-Tudela (2012) where further elements of the derivation are discussed in detail. In particular, U_ϵ, $N_\epsilon(x)$, and $H_\epsilon(x,z)$ are independent of the worker's type, ϵ, and

$$\begin{aligned} H(x,z) \quad &\equiv \quad H^\epsilon(x,z)=\frac{(\phi+\delta)F(z)}{q(z)}\left[1-e^{-q(z)x}\right]- \\ &- \quad \frac{\delta\lambda_u F(z)}{\delta\lambda_u+\lambda_e(1-F(z))(\phi+\lambda_u)}\left[e^{-\frac{\phi(\phi+\delta+\lambda_u)}{\phi+\lambda_u}x}-e^{-q(z)x}\right] \end{aligned} \tag{10}$$

$$N(x)\equiv N^\epsilon(x)=1-\frac{\delta\lambda_u}{(\phi+\lambda_u)(\phi+\delta)}e^{-\frac{\phi(\phi+\delta+\lambda_u)}{\phi+\lambda_u}x} \tag{11}$$

6.2 Value of a vacancy

Upon entering the market a vacancy chooses k (which determines its flow cost during recruitment and affects the product of any future match) and posts z (which affects the probability that a future match will be left by a worker, but also the distribution of experience among workers among whom the firm recruits). Since search is random, a vacancy meets a random worker. If it offers $z < z^R$ no worker accepts; if it offers $z \geq z^R$, all unemployed workers, as well as all workers employed at $z' < z$ accept. Notice that firm's expected turnover is independent of k conditional on z. Then the expected job value for a vacancy offering $\{z, k\}$ is

$$M(z,k) = \int_{\epsilon'} \frac{\lambda_u U^{\epsilon'}}{\lambda_u U^{\epsilon'} + \lambda_e(1-U^{\epsilon'})} \left(\int_{x'=0}^{\infty} J(.,\epsilon',x'|.) dN^{\epsilon'}(x') \right) dA(\epsilon') + \tag{12}$$

$$+ \int_{\epsilon'} \frac{\lambda\lambda_u U^{\epsilon'}}{\lambda_u U^{\epsilon'} + \lambda_e(1-U^{\epsilon'})} \left(\int_{x'=0}^{\infty} \int_{z'\in[z^R,z)} J(.,\epsilon',x'|.) dH^{\epsilon'}(x',z') \right) dA(\epsilon')$$

whenever $z \geq z^R$ and

$$M(z,k) = 0$$

for $z < z^R$.

Given equations (7), (11), (10) and (5) a closed form expression for the expected value of a job for a vacancy posting given $\{z, k\}$ can be obtained. To derive it I conduct the integration in (12) directly (full workings are demonstrated in Appendix A) and after imposing the free entry condition in (6), it follows that for all $z \in [\underline{z}, \bar{z}]$

$$0 = -c(k) + \frac{\eta(\theta)\tilde{\epsilon}(\phi+\delta+\lambda_u)}{(\phi+\delta+\lambda_e)} \left[\frac{p(k)-z}{q(z)+r-\rho} \right] \left[a_0 + \lambda a_1 \frac{F(z)}{q(z)-\rho} \right] \tag{13}$$

where

$$a_0 = \frac{\phi\delta\lambda_u}{(\phi+\lambda_u)(\phi(\phi+\delta+\lambda_u)-\rho(\phi+\lambda_u))}$$

$$a_1 = \frac{\phi(\phi+\delta-\rho)\lambda_u}{(\phi(\phi+\delta+\lambda_u)-\rho(\phi+\lambda_u))}$$

$$\tilde{\epsilon} = \int_\epsilon \epsilon' dA(\epsilon')$$

A firm choosing z faces a tradeoff: a lower z means higher profit flow whoever the firm is matched with; a higher z means lower probability of the worker leaving the firm and also higher expected productivity of the match. For future references, let $a(z) \equiv a_0 + \lambda a_1 F(z)/(q(z)-\rho)$.

6.3 Investment choice

I now turn to the optimal choice of investment, k. Suppose that the pair $\{z^*, k^*\}$ is an optimal policy. Since z^* is optimal, the envelope theorem implies that the optimal choice of k requires (differentiating (13) with respect to k at the optimum)

$$c_k(k^*) = \frac{\eta\tilde{\epsilon}(\phi+\delta+\lambda_u)}{(q(z^*)+r-\rho)(\phi+\delta+\lambda_u)}\left[a_0 + \lambda a_1 \frac{F(z^*)}{q(z^*)-\rho}\right] p_k(k^*) \tag{14}$$

It is easy to see that the restrictions on $p(.)$ and $c(.)$ guarantee that the latter identifies a maximum and the solution is unique. Furthermore, combining (13) and (14) yields

$$z^* = p(k^*) - p_k(k^*)\frac{c(k^*)}{c_k(k^*)} \tag{15}$$

The right-hand side is strictly increasing in k and therefore (15) describes a one-to-one mapping between z and k irrespective of the functional form of $F(.)$. By offering higher pay rates a vacancy expects to match with more productive workers and also to keep them for longer. Since their productivity augments its own, it finds investment in capital more profitable - firms that offer high pay rates invest more in capital. Henceforth, let $k(z)$ denote the optimal choice of investment for a firm offering z, and let $p(z) \equiv p(k(z))$ and $c(z) \equiv c(k(z))$.

6.4 Offer distribution

I now characterise the equilibrium distribution of z. First, notice that all vacancies yield zero value in expectation and in particular the value from offering z^R is the same as from offering the highest equilibrium rate $\bar{z}$. Equating $V(z^R, k(z^R))$ to $V(\bar{z}, k(\bar{z}))$ (assuming non-degenerate distribution) yields an expression for the highest equilibrium offer, $\bar{z}$ in terms of z^R, which can be conveniently expressed as

$$\left(\frac{p(\bar{z}) - \bar{z}}{c(\bar{z})}\right) = \left(\frac{p(z_R) - z_R}{c(z_R)}\right) \frac{\delta(\phi + \delta + r - \rho)}{(\delta + \lambda(\phi + \lambda_u))(\phi + \delta + \lambda_e + r - \rho)} \tag{16}$$

Notice that given (15), the restrictions on $p(.)$ and $c(.)$ guarantee that $(p(z) - z)/c(z)$ is invertible, and for each level of tightness and z_R, (16) determines a unique $\bar{z}$.

Next, equating $V(z, k(z))$ to $V(z^R, k(z^R))$ yields

$$\left(\frac{a_0}{q(z) + r - \rho} + a_1 \frac{\lambda F(z)}{(q(z) + r - \rho)(q(z) - \rho)}\right) =$$

$$= \frac{p(z_R) - z_R}{c(z_R)} \frac{c(z)}{p(z) - z} \frac{a_0}{\phi + \delta + \lambda_e + r - \rho}$$

Denoting the right-hand side by $b_0(z)$ the latter can be restated as the quadratic equation in F

$$b_0\lambda_e^2 F^2 - [b_0(2(\phi+\delta+\lambda_e-\rho)+r)\lambda_e + (\lambda a_1 - \lambda_e a_0)]F +$$

$$(\phi+\delta+\lambda_e-\rho)[b_0(\phi+\delta+\lambda_e+r-\rho)-a_0] = 0$$

with discriminant

$$D = (b_0\lambda_e r + (\lambda a_1 - \lambda_e a_0))^2 + 4b_0\lambda_e(\phi+\delta+\lambda_e-\rho)a_1 > 0$$

and roots

$$F_{1,2} = 1 + \frac{r+2(\phi+\delta-\rho)}{2\lambda_e} + \frac{\phi}{2b_0\lambda_e(\phi+\lambda_u)} \pm \frac{\sqrt{D}}{2b_0\lambda_e\lambda_u} \tag{17}$$

The root consistent with F being distribution is the smaller of the two (the other one exceeds one). The equilibrium offer distribution is therefore determined uniquely given z_R and θ.

6.5 Tightness

Finally, equilibrium tightness is determined by the free entry condition. Evaluating (13) at $z = z^R$, expressing the matching rates in terms of tightness, and discarding the no-trade solution (dividing both sides of the equation by θ^α) yields

$$\theta^{1-2\alpha} = \frac{\phi\delta\beta^2\tilde{\epsilon}[(\phi+\delta+\beta\theta^\alpha)/(\phi+\delta+\lambda\beta\theta^\alpha)][p(z^R)-z^R]/c(z^R)}{(\phi+\delta+\lambda\beta\theta^\alpha+r-\rho)(\phi+\beta\theta^\alpha)(\phi(\phi+\delta+\beta\theta^\alpha)-\rho(\phi+\beta\theta^\alpha))} \tag{18}$$

Notice that the right-hand side is positive, continuously decreasing in θ and approaches zero as θ limits to infinity. The behaviour of the left-hand side expression, however, depends on the sign of $(1-2\alpha)$. It is immediate that $\alpha < 0.5$ is sufficient (but not necessary) for existence of a unique $\theta(z^R) > 0$ and given this the relationship between θ and z_R is negative[8]. Further, notice that as z^R limits to infinity, the solution to (18) limits to zero[9].

6.6 Existence

Equilibrium is now fully characterised. To summarize, given z^R market tightness is determined by (18). Given θ and z_R, (17) determines the offer distribution. Given $F(.)$ and θ, (3) determines z_R. Further, for each $z \in [z_R, \bar{z}]$ the associated optimal capital investment is determined by (15) and the steady-state distributions are given by the results in Section 6.1.

Given this, an equilibrium is a zero of the function $T(z^R)$ defined as

$$T(z^R) \equiv (r+\phi)[z^R - z^R(F(z|z^R), \theta(z^R))] = \tag{19}$$

$$= (r+\phi)z^R - (r+\phi-\rho)b -$$

$$-[\lambda_u(\theta|z^R)(r+\phi-\rho) - \lambda_e(\theta|z^R)(r+\phi)] \int_{z^R}^{\bar{z}(z^R)} \frac{1-F(z'|z^R)}{q(z'|z^R)+r-\rho} dz'$$

where $z^R(F(.), \theta)$ is the solution to (3), $\theta(z^R)$ is the solution to (18) and $F(z|z^R)$ is the solution to (17). Notice that

$$T((r+\phi-\rho)b/(r+\phi)) < 0$$

(as search has option value) and

$$\lim_{z^R \to \infty} T(z^R) = +\infty$$

[8]Establishing a weaker sufficient condition analytically is hindered by the complexity of the expression and is not pursued. In the numerical exercises below I find that a unique solution obtains under every parameterisation used.

[9]As $(p(z^R) - z^R)/c(z^R) = p'(z^R)/c'(z^R)$ limits to zero as long as $\lim_{z \to \infty} p'(z) = 0$.

as (18) implies that θ approaches zero in the limit. As $T(.)$ is continuous, a disperse equilibrium with $z^R > (r+\phi-\rho)b/(r+\phi)$ exists. The latter, however, need not be unique for any values of the parameters. Differentiating (19) with respect to z^R yields

$$
\begin{aligned}
T_{z^R} &= (r+\phi) - \frac{\partial\theta}{\partial z^R}\beta\alpha\theta^{\alpha-1}[(r+\phi-\rho)-\lambda(r+\phi)]\int_{z^R}^{\bar{z}(z^R)}\frac{1-F(z'|z^R)}{q(z'|z^R)+r-\rho}dz' \\
&+ \frac{[(r+\phi-\rho)-\lambda(r+\phi)]\beta\theta^{\alpha}}{\phi+\delta+\lambda_e+r-\rho} \\
&- [(r+\phi-\rho)-\lambda(r+\phi)]\beta\theta^{\alpha}\int_{z^R}^{\bar{z}(z^R)}\frac{\partial}{\partial z^R}\left[\frac{1-F(z'|z^R)}{q(z'|z^R)+r-\rho}\right]dz'
\end{aligned}
$$

While the first three terms of the sum are positive, the last may be negative. This being said, in the numerical exercise that follows unique disperse equilibrium obtains under the parameterisation used.

7 Numerical analysis

This section presents the results from numerically solving the model and, in particular, discusses how equilibrium dispersion depends on the rate of human capital accumulation and the parameters governing investment choice. The discussion centres on the implications for offer and productivity distributions rather than the distribution of piece-rates or wages across employed workers. The equilibrium distribution of offers has the immediate interpretation of the exact counterpart of frictional wage dispersion - workers of the same ability and experience who become employed simultaneously, draw initial wages from this distribution. The overall wage distribution (among all employed or only the newly-employed) generated by this class of models is more complicated as it depends on the distributions of experience among employed (which is endogenous) and on the distribution of abilities in the population (which is primitive). As discussed in Burdett et al. (2011) and Carrillo-Tudela (2012) the wage density generated by the model inherits the shape of the ability distribution, but some features, in particular related to the right tail of the distribution depend crucially on the underlying dispersion.

7.1 Parameterization

I start by assuming that $\beta = 1$ and $\alpha = 0.5$; that is the matching function is given by

$$m(v, U + \lambda(1 - U)) = v^{1/2}(U + \lambda(1 - U))^{1/2}$$

and use this functional form for the calibration of the other parameters[10]. For the search cost and productivity functions I choose the following functional forms[11]

$$\begin{aligned} c(k) &= ck \\ p(k) &= pk^{\gamma} \end{aligned}$$

Setting the time period equal to one month, I choose parameters following Carrillo-Tudela (2012). Accordingly I set $r = 0.0041$, implying annual interest rate of 5 per cent; $\phi = 0.0021$, consistent with average labour-market life of 40 years; $\delta = 0.012$.

In a benchmark specification (necessary to calibrate the other parameters) I set $\lambda = \lambda_e/\lambda_u = 0.038/0.141$, as estimated by Carrillo-Tudela (2012) for a sample of medium-skilled workers in the BHPS (this identifies λ and sets η for the benchmark); $\rho = 0.0020$ as fitted there to match the observed mean-min ratio with the one obtained in the model for the same sample; and $\gamma = 0.4$. What remains is to set b and $c/\tilde{\epsilon}$ (c and $\tilde{\epsilon}$ are not separately identified but the ratio is sufficient for parameterising the model). As discussed in Carrillo-

[10] It turns out that all following results are robust qualitatively to specifying different values of the matching elasticity as long as α is not very close to 1.

[11] In this formulation, the choice of p matters in the determination of the other parameters only as a scale factor. All the results that follow are reported using $p = 1000$.

Tudela (2012) the ratio b/z^R identifies uniquely the mean-min ratio generated by the model. Further, notice that given tightness, $c/\tilde{\epsilon}$ is uniquely determined as a function of z^R by the free entry condition (18). Using these observations, I identify the two parameters as follows. Given any guess for z^R, I set $c/\tilde{\epsilon}$ consistently (given tightness), solve the firms' problem, set b consistently with the mean-min ratio observed by Carrillo-Tudela (2012) and check that the value from the guess is consistent with the reservation z from the workers' problem (3). I iterate until $T(z^R) = 0$[12]. This procedure identifies b and $c/\tilde{\epsilon}$.

Table 1: Benchmark specification

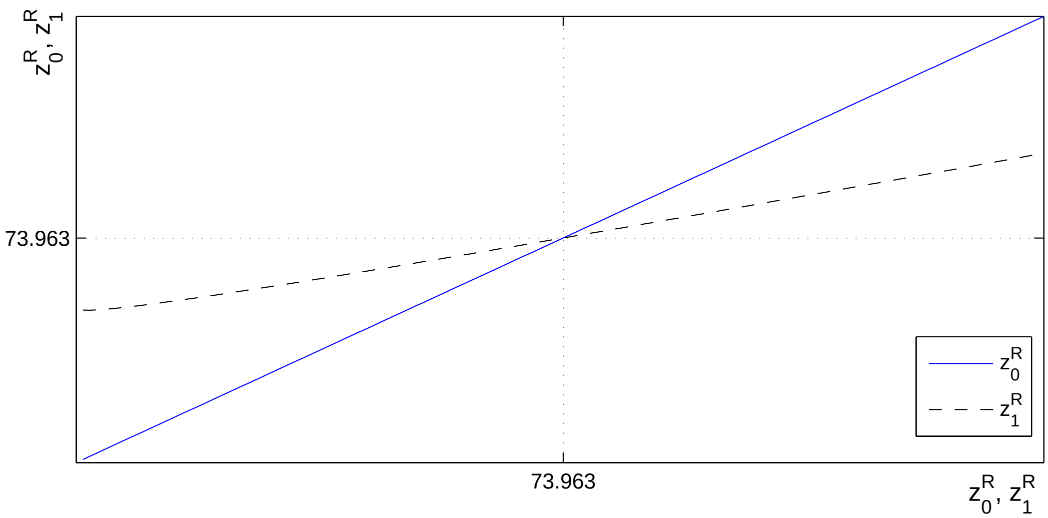
Parameter/variable	Value
Parameterized	
β	1.0000
α	0.5000
λ	0.2695
λ_u	0.1410
λ_e	0.0380
η	7.0920
ρ	0.0019
γ	0.5000
p	1000.0000
Calibrated	
b	448.6623
$c/\tilde{\epsilon}$	6094.0014
Endogenous variables	
$\bar{z}$	1033.0102
z_R	735.9138
$k(\bar{z})$	426.8667
$k(z_R)$	2.1663

7.2 Results

Given the discussed parameterisation, I now present the results from solving the model under different values of ρ and γ. As discussed above uniqueness of equilibrium is not guaranteed

[12]As discussed further this is feasible as for the chosen parameters $T(.)$ behaves like a contraction. See below for further discussion.

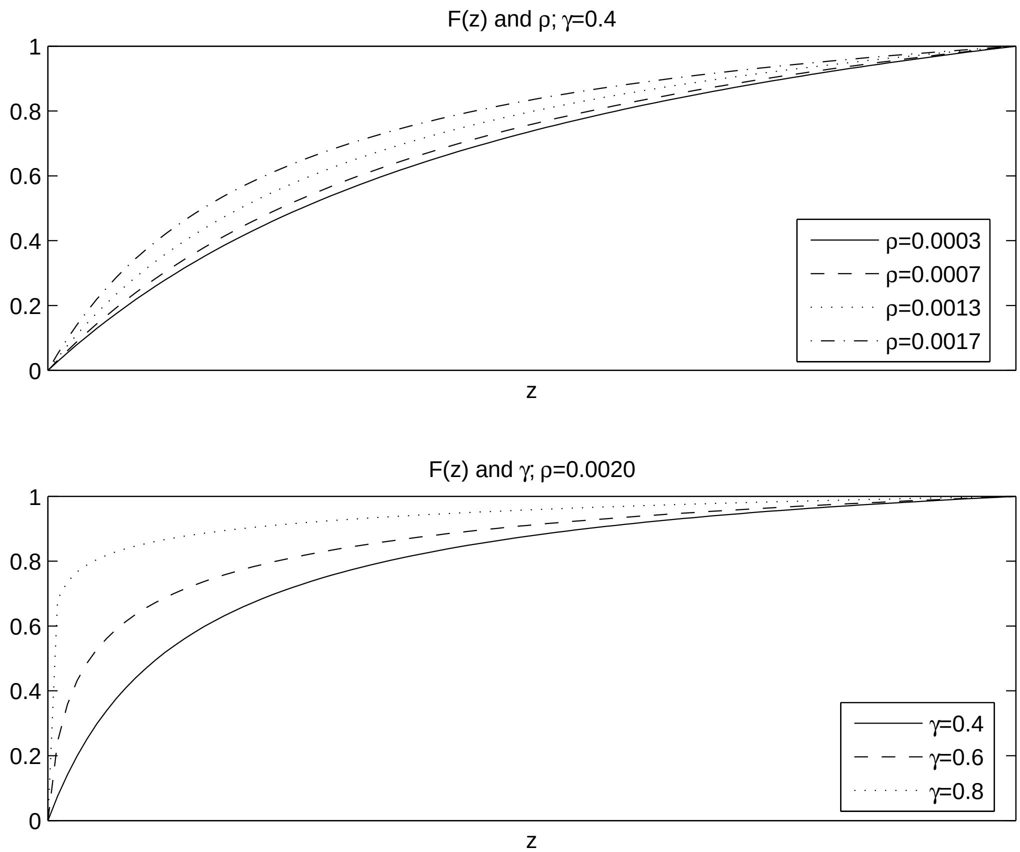
Figure 1: Fixed point for z^R



in general. Under the proposed parameters, however, equilibria are always unique. In particular, consider the following procedure. Starting with a grid for z^R, let z_0^R denote an initial guess. Under the guess the firms' problem can be solved using (18) (identifying θ), (16) (identifying $\bar{z}$ and (17) (identifying $F(.|z_0^R)$). Given these (3) identifies a unique optimal reservation value for workers given firm behaviour. Let z_1^R denote the associated solution. Then z_0^R is an equilibrium if and only if $z_0^R = z_1^R$. Under the benchmark specification ($\rho = 0.0020$ and $\gamma = 0.4$), Figure 1 plots z_0^R and z_1^R for different values of z_0^R. The figure illustrates that a unique fixed point exists and further shows that over the closed interval of z_0^R plotted $T(z^R)$ behaves as a contraction. It turns out that the same applies for different values of ρ and γ. This suggests the feasibility of numerically solving the differently parameterised versions of the model by guessing z^R, updating through (3) and iterating until convergence. This is the method I follow.

To illustrate the relationship between dispersion and the rate of human capital accumulation I set $\gamma = 0.4$ and solve the model for a number of different values of ρ. The top panel of Figure 2 plots the resulting cumulative offer (and productivity) distributions over a normalized support, and the top panel of Figure 3 plots the limits of the support against

Figure 2: Offers distributions, $F(z)$

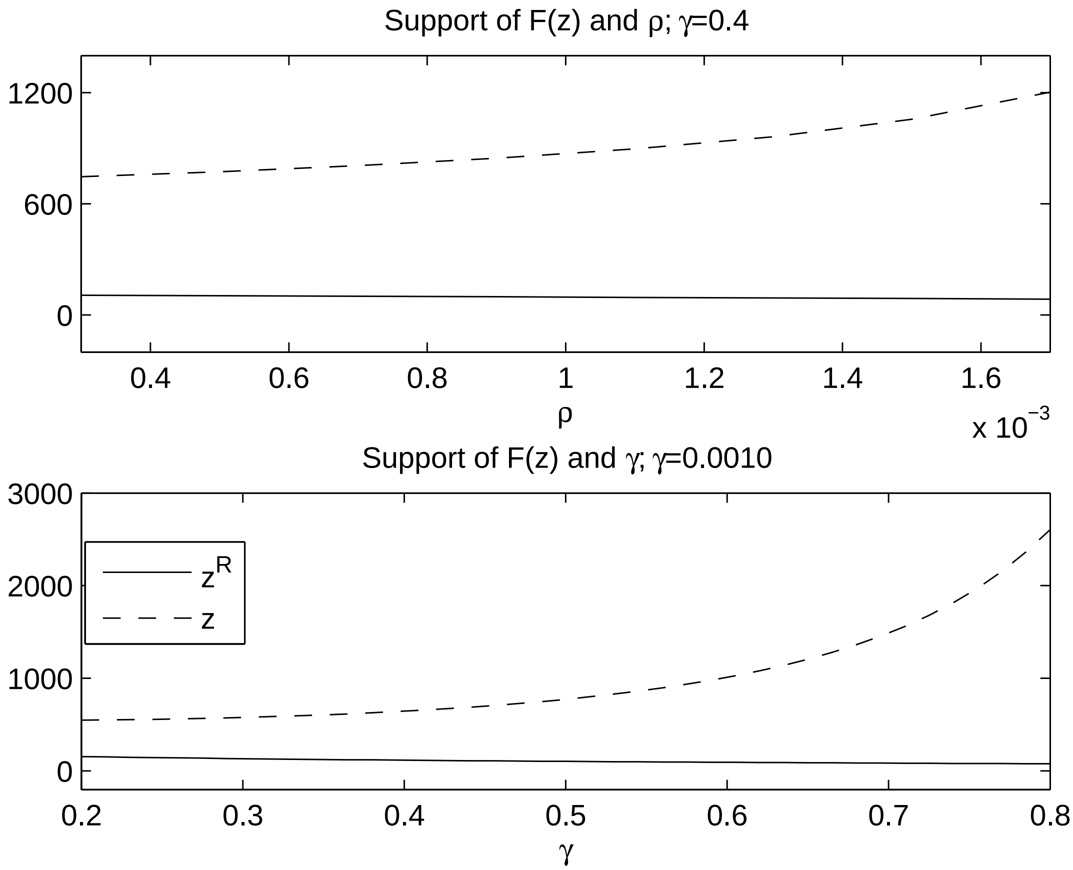


ρ. The results suggest that the distribution gets wider and more right-skewed as the rate of human-capital accumulation increases. Markets characterised by high accumulation are therefore likely to be described by more dispersion both in terms of productivity and wages.

Similarly the bottom panels of Figures 2 and 3 illustrate the relationship between γ and the properties of the distribution. When investment yields larger productivity increases, the resulting steady-state distributions are more unequal. Both the shape and the support of the distributions are highly sensitive and relatively large values of γ can generate extremely unequal distributions.

Two important points should be discussed. First, in all cases the offer densities are continuously decreasing. However, this does not imply that wage densities are. While offer

Figure 3: Support of $F(z)$



distributions are defined over z wages are further determined by individual abilities and histories of experience. For example, if the distribution of abilities is unimodal, then so will be the distribution of wages generated by the model. Second, since each z corresponds to a unique $p(k(z))$, with increasing relationship, the shape of the dispersion of z is always identical to the shape of the distribution of productivities. This is a typical feature of models featuring exogenous productivity dispersion (although here the direction of causality it is not that more productive firms offer higher wages but firms that choose to offer high wages also optimally choose to invest more in productivity). It should again be qualified that the distributions are defined over z rather than wages and the latter also depend on ability and experience. In particular, since more experienced workers sort with more productive firms, the overall wage density (for workers of the same ability) is more unequal than the distribution of firm productivities.

8 Conclusion

This paper presents a model of frictional labour markets where workers accumulate experience while working, search both when unemployed and employed and ex-ante identical firms create vacancies by making costly capital investments in productivity while simultaneously committing to piece rates. In equilibrium, firms with more generous pay policies also invest more and highly experienced (and productive) workers sort with more productive firms. Numerical solutions to the model demonstrate that equilibrium dispersion increases with the rate of human capital accumulation and with the elasticity of productivity with respect to capital investment. By varying the related parameters the model is able to generate different equilibrium distributions including ones that are extremely unequal.

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A Derivation of expected value of a job

This appendix derives the closed form expression for the expected value of a job given $\{z,k\}$, that is

$$M(z,k) = \int_{\epsilon'} \frac{\lambda_u U^{\epsilon'}}{\lambda_u U^{\epsilon'} + \lambda_e(1-U^{\epsilon'})} \left(\int_{x'=0}^{\infty} J(.,\epsilon',x'|.) dN^{\epsilon'}(x') \right) dA(\epsilon') + \tag{20}$$

$$+ \int_{\epsilon'} \frac{\lambda \lambda_u U^{\epsilon'}}{\lambda_u U^{\epsilon'} + \lambda_e(1-U^{\epsilon'})} \left(\int_{x'=0}^{\infty} \int_{z' \in [z^R, z)} J(.,\epsilon',x'|.) dH^{\epsilon'}(x',z') \right) dA(\epsilon')$$

First, notice that by the results from Section 6.1 the unemployment rate and steady state distributions are independent of ϵ. Plugging in (5)

$$\begin{aligned} M(z,k) \quad = \quad & \frac{\lambda_u}{\lambda_u U + \lambda_e(1-U)} \left[\int_{\epsilon'} U \left(\int_{x'=0}^{\infty} \frac{(p(k)-z)\epsilon'}{q(z)+r-\rho} e^{\rho x'} dN(x') \right) dA(\epsilon') + \right. \\ & \left. \lambda(1-U) \int_{\epsilon'} \left(\int_{x'=0}^{\infty} \int_{z' \in [\underline{z}, z)} \frac{(p(k)-z)\epsilon'}{q(z)+r-\rho} e^{\rho x'} dH(x',z') \right) dA(\epsilon') \right] = \end{aligned}$$

$$\begin{aligned} & \frac{\phi+\delta+\lambda_u}{\phi+\delta+\lambda_e} \left[U \frac{(p(k)-z)}{q(z)+r-\rho} \int_{\epsilon'} \epsilon' \left(\int_{x'=0}^{\infty} e^{\rho x'} dN(x') \right) dA(\epsilon') + \right. \\ & \left. \lambda(1-U) \frac{(p(k)-z)}{q(z)+r-\rho} \int_{\epsilon'} \epsilon' \left(\int_{x'=0}^{\infty} e^{\rho x'} \int_{z' \in [\underline{z}, z)} dH(x',z') \right) dA(\epsilon') \right] = \end{aligned}$$

$$\begin{aligned} & \frac{\phi+\delta+\lambda_u}{\phi+\delta+\lambda_e} \frac{(p(k)-z)}{q(z)+r-\rho} \left[U \int_{\epsilon'} \epsilon' dA(\epsilon') \left(\int_{x'=0}^{\infty} e^{\rho x'} dN(x') \right) + \right. \\ & \left. \lambda(1-U) \int_{\epsilon'} \epsilon' dA(\epsilon') \left(\int_{x'=0}^{\infty} e^{\rho x'} H_x(x',z) dx' \right) \right] \end{aligned}$$

The first equality follows as $\frac{\lambda_u}{\lambda_u U + \lambda_e(1-U)} = \frac{\phi+\delta+\lambda_u}{\phi+\delta+\lambda_e}$, $\frac{(p(k)-z)}{q(z)+r-\rho}$ is constant conditional on $\{z,k\}$, and given x', $e^{\rho x'}$ does not vary with z. The second equality follows because turnover is independent of ϵ, and because

$$\int_{z'\in[\underline{z},z)} H_{xz}(x',z')dz' = H_x(x',z) - H_x(x',\underline{z}) = H_x(x',z)$$

Next, differentiating (11) and (10) with respect to x yields the densities

$$\begin{aligned}
\frac{\partial N(x)}{\partial x} &= \frac{\phi(\phi+\delta+\lambda_u)\delta\lambda_u}{(\phi+\lambda_u)^2(\phi+\delta)} e^{-\frac{\phi(\phi+\delta+\lambda_u)}{\phi+\lambda_u}x} \\
\frac{\partial H(x,z)}{\partial x} &= \frac{F(z)\phi(\phi+\delta+\lambda_u)}{\delta\lambda_u + \lambda_e(\phi+\lambda_u)(1-F(z))} \times \\
&\quad \left[\frac{\delta\lambda_u}{\phi+\lambda_u} e^{-\frac{\phi(\phi+\delta+\lambda_u)}{\phi+\lambda_u}} + \lambda_e(1-F(z))e^{-q(z)x}\right]
\end{aligned}$$

Using these above, the expressions for the integrals can be stated as

$$\int_{x'=0}^{\infty} e^{\rho x'} N_x(x')dx' = \frac{\phi\delta\lambda_u(\phi+\delta+\lambda_u)}{(\phi+\lambda_u)^2(\phi+\delta)} \int_{x=0}^{\infty} e^{\left[\rho - \frac{\phi(\phi+\delta+\lambda_u)}{\phi+\lambda_u}\right]x'} dx'$$

and

$$\int_{x'=0}^{\infty} e^{\rho x'} H_x(x',z)dx' = \frac{F(z)\phi(\phi+\delta+\lambda_u)}{\delta\lambda_u + (\phi+\lambda_u)\lambda_e(1-F(z))} \times$$
$$\left[\frac{\delta\lambda_u}{\phi+\lambda_u} \int_{x'=0}^{\infty} e^{\left(\rho - \frac{\phi(\phi+\delta+\lambda_u)}{\phi+\lambda_u}\right)x'} dx' + \lambda_e(1-F(z)) \int_{x'=0}^{\infty} e^{(p-q(z))x'} dx'\right]$$

and direct integration yields

$$\begin{aligned}
\int_{x'=0}^{\infty} e^{\left(\rho - \frac{\phi(\phi+\delta+\lambda_u)}{\phi+\lambda_u}\right)x'} dx' &= \frac{\phi+\lambda_u}{\phi(\phi+\delta+\lambda_u) - \rho(\phi+\lambda_u)} \\
\int_{x'=0}^{\infty} e^{(p-q(z))x'} dx' &= \frac{1}{q(z)-\rho}
\end{aligned}$$

Then after some basic manipulations

$$U \int_{x'=0}^{\infty} e^{\rho x'} dN(x') \quad = \quad a_0$$

and

$$\lambda(1-U) \int_{x'=0}^{\infty} e^{\rho x'} H_x(x', z) dx' \quad = \quad \lambda a_1 \frac{F(z)}{q(z) - \rho}$$

as in (13).